Almost $\mathcal{D}$-split sequences and derived equivalences

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Abstract

In this paper, we introduce almost $\mathcal{D}$-split sequences and establish an elementary but somewhat surprising connection between derived equivalences and Auslander-Reiten sequences via BB-tilting modules. In particular, we obtain derived equivalences from Auslander-Reiten sequences (or n-almost split sequences), and Auslander-Reiten triangles.

1 Introduction

Derived equivalence and Auslander-Reiten sequence are two important objects in the modern representation theory of algebras and groups. On the one hand, derived equivalence preserves many significant invariants of groups and algebras; for example, the number of irreducible representations, Cartan determinants, Hochschild cohomology groups, algebraic K-theory and G-theory(see [7], [11] and [9]). One of the fundamental results on derived categories may be the Morita theory for derived categories established by Rickard in his several papers [20, 21, 22], which says that two rings A and B are derived-equivalent if and only if there is a tilting complex T of A-modules such that B is isomorphic to the endomorphism ring of T. Thus, starting with a ring A, we may construct theoretically all rings which are derived-equivalent to A by finding all tilting complexes of A-modules. However, in practice, it is not easy to show that two given rings are derived-equivalent by finding a suitable tilting complex, as is indicated by the famous unsolved Broue's abelian defect group conjecture, which states that the module categories of a block algebra A of a finite group algebra and its Brauer correspondent B should have equivalent derived categories if their defect groups are abelian (see [7]). On the other hand, as is well-known, Auslander-Reiten sequence is of significant importance in the modern representation theory of Artin algebras, it contains rich combinatorial information on the module category (see [3]). A natural and fundamental question is: Is there any relationship between Auslander-Reiten sequences and derived equivalences ? In other words, is it possible to construct derived equivalences from Auslander-Reiten sequences or n-almost split sequences or Auslander-Reiten triangles ?

In the present paper, we shall provide an affirmative answer to this question and construct derived equivalences by the so-called almost $\mathcal{D}$-split sequences (see Definition 3.1 below). Such sequences include Auslander-Reiten sequences and occur very frequently in the representation theory of Artin algebras (see the examples in Section 3 below). Our result in this direction can be stated in the following general form:

Theorem 1.1 *Let $\mathcal{C}$ be an additive category and M be an object in $\mathcal{C}$. Suppose*

$$X \longrightarrow M' \longrightarrow Y$$

is an almost $\mathrm{add}(M)$*-split sequence in* $\mathcal{C}$*. Then the endomorphism ring* $\mathrm{End}_{\mathcal{C}}(M\oplus X)$ *of* $M\oplus X$ *and the endomorphism ring* $\mathrm{End}_{\mathcal{C}}(M\oplus Y)$ *of* $M\oplus Y$ *are derived-equivalent via a tilting module. Moreover, the finitistic dimension of* $\mathrm{End}_{\mathcal{C}}(M\oplus X)$ *is finite if and only if so is the finitistic dimension of* $\mathrm{End}_{\mathcal{C}}(M\oplus Y)$*.*

This result reveals a mysterious connection between Auslander-Reiten sequences and derived equivalences, namely we have the following corollary.

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Corollary 1.2 *Let A be an Artin algebra.*

(1) *Suppose $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow X_{i-1} \longrightarrow 0$ is an Auslander-Reiten sequence of finitely generated A-modules for $i = 1, 2, \cdots, n$. Let $M = \bigoplus_{i=1}^{n} M_i$. Then $\mathrm{End}_A(M \oplus X_n)$ and $\mathrm{End}_A(M \oplus X_0)$ are derived-equivalent via an n-BB-tilting module. In particular, if $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander-Reiten sequence, then the endomorphism algebras $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$ are derived-equivalent via BB-tilting module, and have the same Cartan determinant.*

(2) *If A is self-injective and X is an A-module, then the endomorphism algebra $\mathrm{End}\,(A \oplus X)$ of $A \oplus X$ and the endomorphism algebra $\mathrm{End}_A(A \oplus \Omega(X))$ of $A \oplus \Omega(X)$ are derived-equivalent, where Ω is the syzygy operator.*

Thus, by Corollary 1.2 or more generally, by Proposition 3.13 in Section 3 below, one can produce a lot of derived equivalences from Auslander-Reiten sequences or n-almost split sequences. We stress that the algebra $\mathrm{End}_A(X \oplus M)$ and the algebra $\mathrm{End}_A(M \oplus Y)$ in Corollary 1.2 may be very different from each other (see the examples in Section 6), though the mesh diagram of the Auslander-Reiten sequence is somehow symmetric. Another result related to Corollary 1.2 is Proposition 5.1 in Section 5 below, which produces derived equivalences from Auslander-Reiten triangles in a triangulated category. In particular, we have

Corollary 1.3 *Let A be a self-injective Artin algebra. Suppose $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander-Reiten sequence such that $\Omega^{-1}(X) \notin \mathrm{add}\,(M \oplus Y)$. Then $\underline{\mathrm{End}}_A(M \oplus X)$ and $\underline{\mathrm{End}}_A(M \oplus Y)$ are derived-equivalent, where $\underline{\mathrm{End}}_A(M)$ denotes the stable endomorphism algebra of an A-module ${}_AM$.*

The paper is organized as follows: In Section 2, we recall briefly some basic notions and a fundamental result of Rickard on derived categories. Our main results, Theorem 1.1, is proved in Section 3, where we also provide several generalizations of Corollary 1.2; among others is a formulation of Corollary 1.2(1) for n-almost split sequences. In section 4, we point out that if an almost $\mathcal{D}$-split sequence is given by an Auslander-Reiten sequence then Theorem 1.1 can be viewed as a "generalized" version of a BB-tilting module. Thus an n-almost split sequence or concatenating n Auslander-Reiten sequences provides us a natural way to get an n-BB-tilting module (for definition, see Section 4). In Section 5, we discuss how to get derived equivalences from Auslander-Reiten triangles in a triangulated category. In particular, Corollary 1.3 is proved in this section. In the last section we present an example to illustrate our main result.

2 Preliminaries

In this section, we recall some basic definitions and results required in our proofs.

Let $\mathcal{C}$ be an additive category. For two morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathcal{C}$, the composition of f with g is written as fg, which is a morphism from X to Z. But for two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ of categories, their composition is denoted by GF. For an object X in $\mathcal{C}$, we denote by $\mathrm{add}\,(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite sums of copies of X.

A complex $X^\bullet$ over $\mathcal{C}$ is a sequence of morphisms d_X^i between objects X^i in $\mathcal{C}$: $\cdots \rightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \rightarrow \cdots$, such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. We write $X^\bullet = (X^i, d_X^i)$. The category of all complexes over $\mathcal{C}$ with the usual complex maps of degree zero is denoted by $\mathscr{C}(\mathcal{C})$. The homotopy and derived categories of complexes over $\mathcal{C}$ are denoted by $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively. The full subcategory of $\mathscr{C}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{C}^b(\mathcal{C})$. Similarly, $\mathscr{K}^b(\mathcal{C})$ and $\mathscr{D}^b(\mathcal{C})$ denote the full subcategories consisting of bounded complexes in $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively.

An object X in a triangulated category $\mathcal{C}$ with a shift functor [1] is called *self-orthogonal* if $\mathrm{Hom}_{\mathcal{C}}(X, X[n]) = 0$ for all integers $n \neq 0$.

Let A be a ring with identity. By A-module we shall mean a left A-module. We denote by A-Mod the category of all A-modules, by A-mod the category of all finitely presented A-modules, and by A-proj (respectively, A-inj) the category of finitely generated projective (respectively, injective) A-modules. Let X be an A-module. If $f : P \longrightarrow X$ is a projective cover of X with P projective, then the kernel of f is called a *syzygy* of X, denoted by $\Omega(X)$. Dually, if $g : X \longrightarrow I$ is an injective envelope with I injective, then the cokernel of g is called a *co-syzygy* of X, denoted by $\Omega^{-1}(X)$. Note that a syzygy or a co-syzygy of an A-module X is determined, up to isomorphism, uniquely by X. Hence we may speak of the syzygy and the co-syzygy of a module.

It is well-known that $\mathscr{K}(A\text{-Mod})$, $\mathscr{K}^b(A\text{-Mod})$, $\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}^b(A\text{-Mod})$ all are triangulated categories. Moreover, it is known that if $X \in \mathscr{K}^b(A\text{-proj})$ or $Y \in \mathscr{K}^b(A\text{-inj})$, then $\mathrm{Hom}_{\mathscr{K}^b(A-\mathrm{Mod})}(X,Z) \simeq \mathrm{Hom}_{\mathscr{D}^b(A-\mathrm{Mod})}(X,Z)$ and $\mathrm{Hom}_{\mathscr{K}^b(A-\mathrm{Mod})}(Z,Y) \simeq \mathrm{Hom}_{\mathscr{D}^b(A\text{-Mod})}(Z,Y)$ for all $Z \in \mathscr{D}^b(A\text{-Mod})$.

For further information on triangulated categories, we refer to [11]. In [20], Rickard proved the following theorem.

Theorem 2.1 *For two rings A and B with identity, the following are equivalent:*
(*a*) $\mathscr{D}^b(A\text{-Mod})$ *and* $\mathscr{D}^b(B\text{-Mod})$ *are equivalent as triangulated categories;*
(*b*) $\mathscr{K}^b(A\text{-proj})$ *and* $\mathscr{K}^b(B\text{-proj})$ *are equivalent as triangulated categories;*
(*c*) $B \simeq \mathrm{End}_{\mathscr{K}^b(A\text{-proj})}(T^\bullet)$, *where $T^\bullet$ is a complex in* $\mathscr{K}^b(A\text{-proj})$ *satisfying*
(1) $T^\bullet$ *is self-orthogonal in* $\mathscr{K}^b(A\text{-proj})$,
(2) $\mathrm{add}\,(T^\bullet)$ *generates* $\mathscr{K}^b(A\text{-proj})$ *as a triangulated category.*

If two rings A and B satisfy the equivalent conditions of Theorem 2.1, then A and B are said to be *derived-equivalent*. A complex $T^\bullet$ in $\mathscr{K}^b(A\text{-proj})$ satisfying the conditions (1) and (2) in Theorem 2.1 is called a *tilting complex* over A. Given a derived equivalence F between A and B, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over A such that $FT^\bullet = B$. This complex $T^\bullet$ is called a tilting complex *associated* to F.

To get derived equivalences, one may use tilting modules. Recall that a module T over a ring A is called a *tilting module* if
(1) T has a finite projective resolution $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$, where each P_i is a finitely generated projective A-module;
(2) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$, and
(3) there is an exact sequence $0 \longrightarrow A \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^m \longrightarrow 0$ of A-modules with each T^i in $\mathrm{add}\,(T)$.

It is well-known that each tilting module supplies a derived equivalence. The following result in [8] is a generalization of a result in [11, Theorem 2.10].

Lemma 2.2 *Let A be a ring, ${}_AT$ a tilting A-module and $B = \mathrm{End}_A(T)$. Then A and B are derived-equivalent. In this case, we say that A and B are derived-equivalent via a tilting module.*

In Theorem 2.1, if both A and B are left coherent rings, that is, rings for which the kernels of any homomorphisms between finitely generated projective modules are finitely generated, then A-mod and B-mod are abelian categories, and the equivalent conditions in Theorem 2.1 are further equivalent to the condition

(*d*) $\mathscr{D}^b(A\text{-mod})$ *and* $\mathscr{D}^b(B\text{-mod})$ *are equivalent as triangulated categories.*

A special class of coherent rings is the class of Artin algebras. Recall that an *Artin R-algebra* over a commutative Artin ring R is an R-algebra A such that A is a finitely generated R-module. For the module category over an Artin algebra, there is the notion of Auslander-Reiten sequence, or equivalently, almost split sequence. It plays an important role in the modern representation theory of algebras and groups. Recall that a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in A-mod is called an *Auslander-Reiten sequence* if
(1) the sequence does not split,
(2) X and Z are indecomposable,
(3) for any morphism $h : V \longrightarrow Z$ in A-mod, which is not a split epimorphism, there is a homomorphism $f' : V \longrightarrow Y$ in A-mod such that $h = f'f$, and
(4) for any morphism $h : X \longrightarrow V$ in A-mod, which is not a split monomorphism, there is a homomorphism $f' : Y \longrightarrow V$ in A-mod such that $h = ff'$.

For an introduction to Auslander-Reiten sequences and representations of Artin algebras, we refer the reader to the excellent book [3].

3 Almost $\mathcal{D}$-split sequences and derived equivalences

In this section, we shall construct derived equivalences from Auslander-Reiten sequences. This builds a linkage between Auslander-Reiten sequences (or n-almost split sequences) and derived equivalences. We start first with a general setting by introducing the notion of almost $\mathcal{D}$-split sequences, which is a slight generalization of Auslander-Reiten sequences, and then use these sequences to construct derived equivalences between the

endomorphism rings of modules involved in almost $\mathcal{D}$-split sequences. In Section 5, we shall consider the question of getting derived equivalences from Auslander-Reiten triangles.

Now we recall some definitions from [4].

Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and X an object in $\mathcal{C}$. A morphism $f : D \longrightarrow X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and the induced map $\mathrm{Hom}_{\mathcal{C}}(-, f)$: $\mathrm{Hom}_{\mathcal{C}}(D', D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(D', X)$ is surjective for every object $D' \in \mathcal{D}$. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is called *right minimal* if any morphism $g : X \longrightarrow X$ with $gf = f$ is an automorphism. A minimal right $\mathcal{D}$-approximation of X is a right $\mathcal{D}$-approximation of X, which is right minimal. Dually, there is the notion of a *left $\mathcal{D}$-approximation* and a *minimal left $\mathcal{D}$-approximation*. The subcategory $\mathcal{D}$ is called *contravariantly* (respectively, *covariantly*) *finite* in $\mathcal{C}$ if every object in $\mathcal{C}$ has a right (respectively, left) $\mathcal{D}$-approximation. The subcategory $\mathcal{D}$ is called *functorially finite* in $\mathcal{C}$ if $\mathcal{D}$ is both contravariantly and covariantly finite in $\mathcal{C}$.

Let $\mathcal{C}$ be an additive category and $e : X \longrightarrow X$ an idempotent morphism in $\mathcal{C}$. We say that e *splits* if there are objects X' and X'' in $\mathcal{C}$ and an isomorphism $\varphi : X' \oplus X'' \longrightarrow X$ such that $\varphi e = \pi\lambda\varphi$, where $\pi : X' \oplus X'' \longrightarrow X'$ and $\lambda : X' \longrightarrow X' \oplus X''$ are the canonical morphisms. In an arbitrary additive category, all idempotents need not split, but of course, in the case where $\mathcal{C}$ is an abelian category, every idempotent splits. If all idempotents in $\mathcal{C}$ split, then so is every full subcategory $\mathcal{D}$ of $\mathcal{C}$ which is closed under direct summands. Moreover, for an additive category $\mathcal{C}$ such that every idempotent splits, we know that, for each object M in $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$ induces an equivalence between $\mathrm{add}(M)$ and $\mathrm{End}_{\mathcal{C}}(M)$-proj.

Definition 3.1 *Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence*

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

in $\mathcal{C}$ is called an almost $\mathcal{D}$-split sequence if

(1) *$M \in \mathcal{D}$;*

(2) *f is a left $\mathcal{D}$-approximation of X, and g is a right $\mathcal{D}$-approximation of Y;*

(3) *f is a kernel of g, and g is a cokernel of f.*

Recall that a morphism $f : Y \longrightarrow X$ in an additive category $\mathcal{C}$ is a *kernel* of a morphism $g : X \longrightarrow Z$ in $\mathcal{C}$ if $fg = 0$, and for any morphism $h : V \longrightarrow X$ in $\mathcal{C}$ with $hg = 0$, there is a unique morphism $h' : V \longrightarrow Y$ such that $h = h'f$. Note that if a morphism has a kernel in $\mathcal{C}$ then it is unique up to isomorphism. A *cokernel* of a given morphism in $\mathcal{C}$ is defined dually. If $f : Y \longrightarrow X$ in $\mathcal{C}$ is a kernel of a morphism $g : X \longrightarrow Z$ in $\mathcal{C}$, then f is a monomorphism, that is, if $h_i : U \longrightarrow Y$ is a morphism in $\mathcal{C}$ for $i = 1, 2$, such that $h_1 f = h_2 f$, then $h_1 = h_2$. Similarly, if $g : X \longrightarrow Z$ in $\mathcal{C}$ is a cokernel of a morphism $f : Y \longrightarrow X$ in $\mathcal{C}$, then g is an epimorphism, that is, if $h_i : Z \longrightarrow V$ is a morphism in $\mathcal{C}$ for $i = 1, 2$, such that $gh_1 = gh_2$, then $h_1 = h_2$.

Notice that an almost $\mathcal{D}$-split sequence may split, whereas an Auslander-Reiten sequence never splits. Now we give some examples of almost $\mathcal{D}$-split sequences.

Examples. (a) Let A be an Artin algebra and $\mathcal{C} = A$-mod. Suppose $\mathcal{D}$ is the full subcategory of A-mod consisting of all projective-injective A-modules in $\mathcal{C}$. If $g : M \longrightarrow X$ is a surjective homomorphism in A-mod with $M \in \mathcal{D}$, then the sequence $0 \longrightarrow \ker(g) \longrightarrow M \longrightarrow X \longrightarrow 0$ is an almost $\mathcal{D}$-split sequence in $\mathcal{C}$, where $\ker(g)$ stands for the kernel of the homomorphism g.

(b) Let A be an Artin algebra and $\mathcal{C} = A$-mod. Suppose $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander-Reiten sequence. Let N be any module such that $M \in \mathrm{add}(N)$, but neither X nor Y belongs to $\mathrm{add}(N)$. If we take $\mathcal{D} = \mathrm{add}(N)$, then the Auslander-Reiten sequence is an almost $\mathcal{D}$-split sequence in $\mathcal{C}$.

(c) Let A be an Artin algebra and $M \in A$-mod. Recall that M is an *almost complete tilting module* if M is a partial tilting module (that is, M has finite projective dimension and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i > 0$), and if the number of all non-isomorphic direct summands of M equals the number of non-isomorphic simple A-modules minus 1. An indecomposable A-module $X \in A$-mod is called a *tilting complement* to M if $M \oplus X$ is a tilting A-module. If an almost complete tilting module M is faithful, then there is an exact (not necessarily infinite) sequence

$$0 \longrightarrow X_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots$$

of A-modules such that $M_i \in \mathrm{add}(M)$. Moreover, if we define $X_i = \mathrm{coker}(f_i)$, the co-kernel of f_i for $i \geq 1$, then $X_i \not\simeq X_j$ for $i \neq j$, $\mathrm{proj.dim}_A(X_i) \geq i$ for any i, and $\{X_i \mid i \geq 0\}$ is a complete set of non-isomorphic indecomposable tilting complements to M. In addition, each $X_i \longrightarrow M_{i+1}$ is a minimal left $\mathrm{add}(M)$-approximation of X_i and each $M_j \longrightarrow X_j$ is a minimal right $\mathrm{add}(M)$-approximation of X_j. Thus the sequence $0 \longrightarrow X_i \longrightarrow M_{i+1} \longrightarrow X_{i+1} \longrightarrow 0$ is an almost $\mathrm{add}(M)$-split sequence in A-mod for all $i \geq 0$.

For further information on almost complete tilting modules and relationship with the generalized Nakayama conjecture, we refer the reader to [6] and [13].

Now we consider some properties of an almost $\mathcal{D}$-split sequence.

Proposition 3.2 *Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$.*

(1) *Suppose $\mathcal{D}'$ is a full subcategory of $\mathcal{D}$. If a sequence $X \longrightarrow M \longrightarrow Y$ in $\mathcal{C}$ is an almost $\mathcal{D}$-split sequence with $M \in \mathcal{D}'$, then it is an almost $\mathcal{D}'$-split sequence in $\mathcal{C}$.*

(2) *If $X \longrightarrow M \xrightarrow{g} Y$ and $X' \longrightarrow M' \xrightarrow{g'} Y'$ are almost $\mathcal{D}$-split sequences in $\mathcal{C}$ such that both g and g' are right minimal, then $Y \simeq Y'$ if and only if the two sequences are isomorphic. Similarly, If $X \xrightarrow{f} M \longrightarrow Y$ and $X' \xrightarrow{f'} M' \longrightarrow Y'$ are almost $\mathcal{D}$-split sequences in $\mathcal{C}$ such that both f and f' are left minimal, then $X \simeq X'$ if and only if the two sequences are isomorphic.*

Proof. (1) is clear. We prove the first statement of (2). If the two sequences are isomorphic, then $X \simeq X'$ and $Y \simeq Y'$. Now assume that $\phi : Y \longrightarrow Y'$ is an isomorphism. Then $g\phi$ factors through g' since g' is a right $\mathcal{D}$-approximation of Y', and we may write $g\phi = hg'$ for some $h : M \longrightarrow M'$. Similarly, there is a homomorphism $h' : M' \longrightarrow M$ such that $g'\phi^{-1} = h'g$. Thus $hh'g = hg'\phi^{-1} = g\phi\phi^{-1} = g$ and $h'hg' = h'g\phi = g'\phi^{-1}\phi = g'$. Since both g and g' are right minimal, the morphisms hh' and $h'h$ are isomorphisms. It follows easily that h itself is an isomorphism. Since f' is a kernel of g' and since f is a kernel of g, there is a morphism $k : X \longrightarrow X'$ and a morphism $k' : X' \longrightarrow X$ such that $kf' = fh$ and $k'f = f'h^{-1}$. Thus $kk'f = kf'h^{-1} = fhh^{-1} = f$. It follows that $kk' = 1_X$ since f is a monomorphism. Similarly, we have $k'k = 1_{X'}$. Hence k is an isomorphism and the two sequences are isomorphic. Similarly, the other statements in (2) can be proved. $\square$

To get an almost $\mathcal{D}$-split sequence, we may use the following proposition. First, we introduce some notations. Let $\mathcal{D}$ be a full subcategory of a category $\mathcal{C}$. An object C in $\mathcal{C}$ is said to be *generated* (respectively, *co-generated*) by $\mathcal{D}$ if there is an epimorphism $D \longrightarrow C$ (respectively, monomorphism $C \longrightarrow D$) with $D \in \mathcal{D}$. We denote by $\mathscr{F}(\mathcal{D})$ the full subcategory of $\mathcal{C}$ consisting of all objects $C \in \mathcal{C}$ generated by $\mathcal{D}$, and by $\mathscr{S}(\mathcal{D})$ the full subcategory of $\mathcal{C}$ consisting of all objects $C \in \mathcal{C}$ co-generated by $\mathcal{D}$.

Proposition 3.3 *Suppose A is a ring with identity and $\mathcal{C} = A$-Mod. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. We define $\mathscr{X}(\mathcal{D}) = \{X \in \mathcal{C} \mid \mathrm{Ext}^1_A(X, \mathcal{D}) = 0\}$ and $\mathscr{Y}(\mathcal{D}) = \{Y \in \mathcal{C} \mid \mathrm{Ext}^1_A(\mathcal{D}, Y) = 0\}$.*

(1) *If $\mathcal{D}$ is contravariantly finite in $\mathcal{C}$, then, for any A-module $Y \in \mathscr{F}(\mathcal{D}) \cap \mathscr{X}(\mathcal{D})$, there is an almost $\mathcal{D}$-split sequence $0 \longrightarrow X \longrightarrow D \longrightarrow Y \longrightarrow 0$ in $\mathcal{C}$.*

(2) *If $\mathcal{D}$ is covariantly finite in $\mathcal{C}$, then, for any A-module $X \in \mathscr{S}(\mathcal{D}) \cap \mathscr{Y}(\mathcal{D})$, there is an almost $\mathcal{D}$-split sequence $0 \longrightarrow X \longrightarrow D \longrightarrow Y \longrightarrow 0$ in $\mathcal{C}$.*

Proof. (1) Since Y is generated by $\mathcal{D}$, there is a surjective right $\mathcal{D}$-approximation of Y, say $g : M \longrightarrow Y$ with $M \in \mathcal{D}$. Let X be the kernel of g. Then it follows from the exact sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ that the sequence $\mathrm{Hom}_A(M, D') \longrightarrow \mathrm{Hom}_A(X, D') \longrightarrow 0$ is exact since $Y \in \mathscr{X}(\mathcal{D})$. This implies that the homomorphism $X \longrightarrow M$ is a left $\mathcal{D}$-approximation of X. Thus we get an almost $\mathcal{D}$-split sequence in $\mathcal{C}$. (2) can be proved analogously. $\square$

Our main purpose of introducing almost $\mathcal{D}$-split sequences is to construct derived equivalences between the endomorphism algebras of objects appearing in almost $\mathcal{D}$-split sequences. The following lemma is useful in our discussions.

Lemma 3.4 *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t} M_1 \xrightarrow{g} Y$$

is a (not necessarily exact) sequence of morphisms in $\mathcal{C}$ with $M_i \in \mathrm{add}\,(M)$ satisfying the following conditions:

(1) *The morphism $f : X \longrightarrow M_n$ is a left $\mathrm{add}\,(M)$-approximation of X, and the morphism $g : M_1 \longrightarrow Y$ is a right $\mathrm{add}\,(M)$-approximation of Y;*

(2) *Put $V = M \oplus X$ and $W = M \oplus Y$. There are two induced exact sequences*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(V, M_n) \rightarrow \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_1) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(V, Y),$$

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(M_1, W) \rightarrow \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}}(M_n, W) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X, W).$$

Then the endomorphism rings $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ are derived-equivalent via a tilting module of projective dimension at most n.

Proof. Let Λ be the endomorphism ring of V, and let T be the cokernel of the map $[t\ 0]_* : \mathrm{Hom}_{\mathcal{C}}(V, M_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M_1 \oplus M)$. Then, by (2), we have an exact sequence of Λ-modules:

$$0 \to \mathrm{Hom}_{\mathcal{C}}(V,X) \to \mathrm{Hom}_{\mathcal{C}}(V,M_n) \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(V,M_2) \to \mathrm{Hom}_{\mathcal{C}}(V,M_1\oplus M) \to T \to 0. \quad (*)$$

Note that all the Λ-modules appearing in the above exact sequence are finitely generated. Applying $\mathrm{Hom}_{\Lambda}(-, \mathrm{Hom}_{\mathcal{C}}(V,M))$ to this sequence, we get a sequence which is isomorphic to the following sequence

$$0 \longrightarrow \mathrm{Hom}_{\Lambda}(T, \mathrm{Hom}_{\mathcal{C}}(V,M)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_1\oplus M, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_2, M) \longrightarrow$$

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_n, M) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X,M) \longrightarrow 0.$$

By the second exact sequence in (2) and the fact that f is a left $\mathrm{add}\,(M)$-approximation of X, we see that this sequence is exact. It follows that $\mathrm{Ext}^i_{\Lambda}(T, \mathrm{Hom}_{\mathcal{C}}(V,M)) = 0$ for all $i > 0$. Hence $\mathrm{Ext}^i_{\Lambda}(T, \mathrm{Hom}_{\mathcal{C}}(V,M')) = 0$ for all $i > 0$ and $M' \in \mathrm{add}\,(M)$. Thus, by applying $\mathrm{Hom}_{\Lambda}(T,-)$ to the exact sequence $(*)$, we get $\mathrm{Ext}^i_{\Lambda}(T,T) \simeq \mathrm{Ext}^{i+n}_{\Lambda}(T, \mathrm{Hom}_{\mathcal{C}}(V,X))$ for all $i > 0$. But $\mathrm{Ext}^{i+n}_{\Lambda}(T, \mathrm{Hom}_{\mathcal{C}}(V,X)) = 0$ for all $i > 0$ since the projective dimension of T is at most n. Thus $\mathrm{Ext}^i_{\Lambda}(T,T) = 0$ for all $i > 0$. Also, it follows from the exact sequence $(*)$ that the following sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(V, X\oplus M) \to \mathrm{Hom}_{\mathcal{C}}(V, M_n\oplus M) \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(V,M_2) \to \mathrm{Hom}_{\mathcal{C}}(V, M_1\oplus M) \to T \to 0$$

is exact, where $\mathrm{Hom}_{\mathcal{C}}(V, X\oplus M)$ is just Λ and the other terms are in $\mathrm{add}\,(T)$. Thus T is a tilting Λ-module of projective dimension at most n.

Next, we show that $\mathrm{End}_{\Lambda}(T)$ and $\mathrm{End}_{\mathcal{C}}(W)$ are isomorphic. If $n = 1$, we set $V' = X$ and $a = [f, 0] : V' \longrightarrow M_1 \oplus M$. For $n \geq 2$, we set $V' = M_2$ and $a = [t, 0] : V' \longrightarrow M_1 \oplus M$. Let $u : V' \longrightarrow V'$ and $v : M_1\oplus M \longrightarrow M_1 \oplus M$ be two morphisms in $\mathcal{C}$. The morphism pair (u, v) is an endomorphism of the sequence $V' \longrightarrow M_1 \oplus M$ if $ua = av$. Let E be the endomorphism ring of the sequence $V' \longrightarrow M_1 \oplus M$. Let I be the subset of E consisting of those endomorphisms (u, v) such that there exists $h : M_1 \oplus M \longrightarrow V'$ with $ha = v$. It is easy to check that I is an ideal of E. We shall show that $\mathrm{End}_{\mathcal{C}}(W)$ is isomorphic to the quotient ring E/I. Let b be the morphism $\left[\begin{smallmatrix} g & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right] : M_1 \oplus M \longrightarrow W$. Then, by the second exact sequence of the condition (2), we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W,W) \xrightarrow{b^*} \mathrm{Hom}_{\mathcal{C}}(M_1\oplus M, W) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{C}}(V', W). \qquad (**)$$

By considering the image of id_W under the composition b^*a^*, we have $ab = 0$. Thus, for each $(u, v) \in E$, we have $avb = uab = 0$, which means that vb is in the kernel of a^*. Therefore, there is a unique map $q : W \to W$ such that $bq = vb$. Now, we define $\eta : E \to \mathrm{End}_{\mathcal{C}}(W)$ by sending (u, v) to q. Then η is clearly a ring homomorphism. We claim that η is surjective. Indeed, since g is a right $\mathrm{add}\,(M)$-approximation of Y, it is easy to check that the map b is a right $\mathrm{add}\,(M)$-approximation of W. Let q be an endomorphism of W. Then there is a morphism $v : M_1 \oplus M \longrightarrow M_1 \oplus M$ such that $vb = bq$. By the first exact sequence in (2), we have the following exact sequence:

$$\mathrm{Hom}_{\mathcal{C}}(V', V') \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{C}}(V', M_1\oplus M) \xrightarrow{b_*} \mathrm{Hom}_{\mathcal{C}}(V', W).$$

It follows from $avb = abq = 0$ that av is in the kernel of b_* and there is a map $u : V' \longrightarrow V'$ such that $ua = av$. This implies that (u, v) is in E and $\eta(u, v) = q$. Hence η is surjective.

Now, we determine the kernel of η. Note that, by the first exact sequence in (2), we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M_1\oplus M, V') \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{C}}(M_1\oplus M, M_1\oplus M) \xrightarrow{b_*} \mathrm{Hom}_{\mathcal{C}}(M_1\oplus M, W).$$

Now, suppose (u, v) is in the kernel of η. Then $vb = 0$, which means that v is in the kernel of b_*. Hence there is a map $h : M_1 \oplus M \longrightarrow V'$ such that $ha = v$. This implies $(u, v) \in I$. On the other hand, if $(u, v) \in I$ and if η sends (u, v) to q, then $bq = vb = hab = 0$ and q is in the kernel of b^*. By the exact sequence $(**)$, we have $q = 0$. Hence I is the kernel of η, and therefore $E/I \simeq \mathrm{End}_{\mathcal{C}}(W)$.

Let $\overline{E}$ be the endomorphism ring of the following complex of Λ-modules:

$$\mathrm{Hom}_{\mathcal{C}}(V, V') \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{C}}(V, M_1\oplus M),$$

and $\overline{I}$ the ideal of $\overline{E}$ consisting of those $(\overline{u}, \overline{v})$ such that $\overline{h}a_* = \overline{v}$ for some $\overline{h} : \mathrm{Hom}_{\mathcal{C}}(V, M_1\oplus M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, V')$. Similarly, we can show that $\mathrm{End}_{\Lambda}(T)$ is isomorphic to $\overline{E}/\overline{I}$. Finally, the natural map $e : E \longrightarrow$

$\overline{E}$, which sends (u, v) to (u_*, v_*), is clearly an isomorphism of rings and induces an isomorphism from the ring E/I to the ring $\overline{E}/\overline{I}$. Thus $\mathrm{End}_\Lambda(T)$ and $\mathrm{End}_{\mathcal{C}}(W)$ are isomorphic. The proof is completed. $\square$

Remarks. (1) For an Auslander-Reiten sequence $0 \to X \to M \to Y \to 0$ in A-mod with A an Artin algebra, the proof that $\mathrm{End}(_AT)$ of the tilting module T defined in Lemma 3.4 is isomorphic to $\mathrm{End}_A(M \oplus Y)$ can be carried out very easily.

(2) From the proof of Lemma 3.4 we see that if we replace the second exact sequence in (2) by the following two exact sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, M) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(M_1, M) \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(M_n, M) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X, M),$$

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, Y) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(M_1, Y) \xrightarrow{t^*} \mathrm{Hom}_{\mathcal{C}}(M_2, Y),$$

then Lemma 3.4 still holds true. (Here $M_2 = X$ if $n = 1$.) However, in most of cases that we are interested in, the second exact sequence in (2) does exist.

(3) A special case of Lemma 3.4 is the n-almost split sequences in a maximal $(n-1)$-orthogonal subcategory studied in [16]. Let A be a finite-dimensional algebra over a field. Suppose $\mathcal{C}$ is a functorially finite and full subcategory of A-mod. Recall that $\mathcal{C}$ is called a *maximal $(n-1)$-orthogonal* subcategory if $\mathrm{Ext}^i_A(X, Y) = 0$ for all $X, Y \in \mathcal{C}$ and all $0 < i \leq n-1$, and $\mathcal{C} = \mathcal{C} \cap \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(C, X) = 0 \text{ for } C \in \mathcal{C} \text{ and } 0 < i \leq n-1\}$ $= \mathcal{C} \cap \{Y \in A\text{-mod} \mid \mathrm{Ext}^i_A(Y, C) = 0 \text{ for } C \in \mathcal{C} \text{ and } 0 < i \leq n-1\}$. In [16]. It is shown that, for any non-projective indecomposable X in $\mathcal{C}$ (respectively, non-injective indecomposable Y in $\mathcal{C}$), there is an exact sequence

$$(*) \quad 0 \to Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0$$

with $C_j \in \mathcal{C}$ and f_j being radical maps such that the following induced sequences are exact on $\mathcal{C}$:

$$0 \longrightarrow \mathcal{C}(-, Y) \longrightarrow \mathcal{C}(-, C_{n-1}) \longrightarrow \cdots \longrightarrow \mathcal{C}(-, C_0) \longrightarrow \mathrm{rad}_{\mathcal{C}}(-, X) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}(X, -) \longrightarrow \mathcal{C}(C_0, -) \longrightarrow \cdots \longrightarrow \mathcal{C}(C_{n-1}, -) \longrightarrow \mathrm{rad}_{\mathcal{C}}(Y, -) \longrightarrow 0,$$

where $\mathrm{rad}_{\mathcal{C}}$ stands for the Jacobson radical of the category $\mathcal{C}$. Note also that f_0 is a minimal right almost split morphism and that f_n is a minimal left almost split morphism. The sequence $(*)$ is called an n-almost split sequence in [16].

With Lemma 3.4 in mind, now we can show the significance of an almost $\mathcal{D}$-split sequence for constructing derived equivalences by the following result.

Theorem 3.5 *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an almost add(M)-split sequence in $\mathcal{C}$. Then the endomorphism ring $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ of $M \oplus X$ and the endomorphism ring $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ of $M \oplus Y$ are derived-equivalent.

Proof. Let $V = M \oplus X$ and $W = M \oplus Y$. We shall verify the conditions of Lemma 3.4 for $n = 1$. By the definition of an almost $\mathcal{D}$-split sequence, we see immediately that the condition (1) in Lemma 3.4 is satisfied, while the condition (2) in Lemma 3.4 is implied by the condition (3) in Definition 3.1: In fact, by applying $\mathrm{Hom}_{\mathcal{C}}(V, -)$ to the above sequence, we get a complex of abelian groups

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \xrightarrow{(-,f)} \mathrm{Hom}_{\mathcal{C}}(V, M') \xrightarrow{(-,g)} \mathrm{Hom}_{\mathcal{C}}(V, Y).$$

Since f is a monomorphism, the map $(-, f)$ is injective. Clearly, the image of the map $(-, f)$ is contained in the kernel of the map $(-, g)$. Since f is a kernel of g, it is easy to see that the kernel of $(-, g)$ is equal to the image of $(-, f)$. Thus $(*)$ is exact. Similarly, we see that the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{(g,-)} \mathrm{Hom}_{\mathcal{C}}(M', W) \xrightarrow{(f,-)} \mathrm{Hom}_{\mathcal{C}}(X, W)$$

is exact. Thus Theorem 3.5 follows from Lemma 3.4 if we take $n = 1$. $\square$

In Theorem 3.5, the two rings $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ are linked by a tilting module of projective dimension at most 1. This is precisely the case of classic tilting module. Thus there is a nice linkage between

the torsion theory defined by the tilting module in $\mathrm{End}_{\mathcal{C}}(M \oplus X)$-mod and the one in $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$-mod. For more details we refer to [5] and [12].

In the following, we deduce some consequences of Theorem 3.5. Since an Auslander-Reiten sequence can be viewed as an almost $\mathcal{D}$-split sequence, as explained in Example (b), we have the following corollary.

Corollary 3.6 *Let A be an Artin algebra, and let $0 \to X \to M \to Y \to 0$ be an Auslander-Reiten sequence in A-mod. Suppose N is an A-module in A-mod such that neither X nor Y belongs to $\mathrm{add}(N)$. Then $\mathrm{End}_A(N \oplus M \oplus X)$ is derived-equivalent to $\mathrm{End}_A(N \oplus M \oplus Y)$. In particular, $\mathrm{End}_A(M \oplus X)$ and $\mathrm{End}_A(M \oplus Y)$ are derived-equivalent.*

As another consequence of Theorem 3.5, we have the following corollary.

Corollary 3.7 *Let A be an Artin algebra and X a torsion-less A-module, that is, X is a submodule of a projective module in A-mod. If $f : X \to P$ is a left $\mathrm{add}({}_AA)$-approximation of X, then $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}({}_AA \oplus \mathrm{coker}(f))$ are derived-equivalent. In particular, if A is a self-injective Artin algebra, then, for any X in A-mod, the algebras $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \Omega(X))$ are derived-equivalent via a tilting module.*

Proof. Note that f is injective. Thus the short exact sequence

$$0 \longrightarrow X \xrightarrow{f} P \longrightarrow \mathrm{coker}(f) \longrightarrow 0$$

is an almost $\mathrm{add}({}_AA)$-split sequence in A-mod. By Theorem 3.5, the corollary follows. $\square$

As a consequence of Corollary 3.7, we get the following corollary.

Corollary 3.8 *Let A be a self-injective Artin algebra and X an A-module. Then the algebras $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \tau X)$ are derived-equivalent, where τ stands for the Auslander-Reiten translation. Thus, for all n, the algebras $\mathrm{End}_A(A \oplus \tau^n X)$ are derived-equivalent.*

Proof. Let ν be the Nakayama functor $D\mathrm{Hom}_A(-, A)$. It is known that if A is self-injective then $\tau \simeq \nu\Omega^2$, $\nu(A) = A$ and the Nakayama functor is an equivalence from A-mod to itself. Since the algebra $\mathrm{End}_A(A \oplus \tau X)$ is isomorphic to the algebra $\mathrm{End}_A(A \oplus \Omega^2(X))$, the corollary follows immediately from Corollary 3.7. $\square$

Remark. If A is a finite-dimensional self-injective algebra, then, for any A-module X, it was shown in [19, Corollary 1.2] that the algebras $\mathrm{End}_A(A \oplus X)$, $\mathrm{End}_A(A \oplus \Omega(X))$ and $\mathrm{End}_A(A \oplus \tau X)$ are stably equivalent of Morita type. Thus they are both derived-equivalent and stably equivalent of Morita type. For further information on stably equivalences of Morita type for general finite-dimensional algebras, we refer the reader to [17, 18, 19, 24] and the references therein.

Now, we point out the following consequence of Theorem 3.5: if $0 \to X \to M' \to Y \to 0$ is an almost $\mathcal{D}$-split sequence in A-mod with $\mathcal{D} = \mathrm{add}(M)$ for an A-module M, then X and Y have the same number of non-isomorphic indecomposable direct summands which are not in $\mathrm{add}(M)$. This follows from the fact that a derived equivalence preserves the number of non-isomorphic simple modules.

Many other invariants of derived equivalences can be used to study the algebras $\mathrm{End}_A(M \oplus X)$ and $\mathrm{End}_A(M \oplus Y)$; for example, $\mathrm{End}_A(M \oplus X)$ has finite global dimension if and only if $\mathrm{End}_A(M \oplus Y)$ has finite global dimension. This follows from the fact that derived equivalence preserves the finiteness of global dimension. In fact, we have the following explicit formula by tilting theory (see [12] and [11, Proposition 3.4, p.116], for example):

If $0 \to X \to M' \to Y \to 0$ is an almost $\mathcal{D}$-split sequence in A-mod with $\mathcal{D} = \mathrm{add}(M)$ for an A-module M in A-mod, then

$$\mathrm{gl.dim}(\mathrm{End}_{\mathcal{C}}(M \oplus X)) - 1 \leq \mathrm{gl.dim}(\mathrm{End}_{\mathcal{C}}(M \oplus Y)) \leq \mathrm{gl.dim}(\mathrm{End}_{\mathcal{C}}(M \oplus X)) + 1,$$

where $\mathrm{gl.dim}(A)$ stands for the global dimension of A. Note that the global dimension of $\mathrm{End}_{\mathcal{C}}(M \oplus X))$ may be infinite (see Example 3 in Section 6). Concerning global dimensions and Auslander-Reiten sequences, there is a related result which can be found in [14].

Note that if a derived equivalence between two rings A and B is obtained from a tilting module ${}_AT$, that is, there exists a tilting A-module ${}_AT$ such that $B \simeq \mathrm{End}_A(T)$, then the finitistic dimension of A is finite if and only if the finitistic dimension of B is finite (see [10])[a]. Recall that the *finitistic dimension* of an Artin algebra

[a]Recently, it is shown that the finiteness of finitistic dimension is invariant under an arbitrary derived equivalence.

A, denoted by fin.dim(A), is defined to be the supremum of the projective dimensions of finitely generated A-modules of finite projective dimension. The finitistic dimension conjecture states that fin.dim(A) should be finite for any Artin algebra A. This conjecture has closely been related to many other homological conjectures in the representation theory of algebras. For some advances and further information on the finitistic dimension conjecture, we may refer the reader to the recent paper [25] and the references therein.

Thus we have the following corollary.

Corollary 3.9 *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an almost add(M)*-split sequence in $\mathcal{C}$. Then the finitistic dimension of* $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ *is finite if and only if the finitistic dimension of* $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ *is finite.*

If A is an Artin R-algebra over a commutative Artin ring R and M is an A-bimodule, then $A \ltimes M$, the trivial extension of A by M is the R-algebra whose underlying R-module is $A \oplus M$, with multiplication given by

$$(\lambda, m)(\lambda', m') = (\lambda\lambda', \lambda m' + m\lambda')$$

for $\lambda, \lambda' \in A$, and $m, m' \in M$. It is shown in [21] that if A and B are finite-dimensional algebras over a field k that are derived-equivalent, then $A \ltimes D(A)$ is derived-equivalent to $B \ltimes D(B)$, where $D = \mathrm{Hom}_k(-, k)$. Note that $A \ltimes D(A)$ is a self-injective algebra and that a derived equivalence between two self-injective algebras implies a stable equivalence of Morita type between them by [21]. It is known in [23] that a stable equivalence of Morita type preserves representation dimension (see [2] for definition). Hence we have the following corollary.

Corollary 3.10 *Let Λ be a finite-dimensional algebra over a field k and M a Λ-module in* Λ-mod*. Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an almost add(M)*-split sequence in* Λ-mod*, and let $A = \mathrm{End}_{\Lambda}(X \oplus M)$ and $B = \mathrm{End}_{\Lambda}(M \oplus Y)$. Then $A \ltimes D(A)$ is derived-equivalent to $B \ltimes D(B)$. In particular, the representation dimensions of $A \ltimes D(A)$ and $B \ltimes D(B)$ are equal.*

In the following, we consider several generalizations of Corollary 3.6, namely we deal with the case of a finite family of Auslander-Reiten sequences.

Corollary 3.11 *Let A be an Artin algebra, and let $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow X_{i-1} \longrightarrow 0$ be an Auslander-Reiten sequence in* A-mod *for $i = 1, 2, \cdots, n$. Let $M = \bigoplus_{i=1}^n M_i$. Then* $\mathrm{End}_A(M \oplus X_n)$ *and* $\mathrm{End}_A(M \oplus X_0)$ *are derived-equivalent via a tilting module T of projective dimension at most n.*

Proof. First, we suppose $X_n \in \mathrm{add}(M)$. Then there is an M_i such that X_n is a direct summand of M_i, and therefore there is an irreducible map from X_i to X_n. It follows that there is an irreducible map from $X_0 = \tau^{-i}X_i$ to $X_{n-i} = \tau^{-i}X_n$, where τ stands for the Auslander-Reiten translation. Thus X_0 is a direct summand of M_{n-i+1}, which implies $X_0 \in \mathrm{add}(M)$. Hence $\mathrm{add}(M \oplus X_n) = \mathrm{add}(M) = \mathrm{add}(M \oplus X_0)$. Consequently, the algebras $\mathrm{End}_A(M \oplus X_n)$ and $\mathrm{End}_A(M \oplus X_0)$ are Morita equivalent. Thus $\mathrm{End}_A(M \oplus X_n)$ and $\mathrm{End}_A(M \oplus X_0)$ are, of course, derived-equivalent via a (projective) tilting module.

Next, we assume $X_n \notin \mathrm{add}(M)$. In this case, we claim that there is no integer $i \in \{0, 1, \cdots, n\}$ such that $X_i \in \mathrm{add}(M)$. If $X_0 \in \mathrm{add}(M)$, then there is an $M_i, 1 \leq i \leq n$, such that X_0 is a direct summand of M_i. Thus there is an irreducible map from X_i to X_0. By applying the Auslander-Reiten translation, we see that there is an irreducible map from $X_n = \tau^{n-i}X_i$ to $X_{n-i} = \tau^{n-i}X_0$. Hence X_n is a direct summand of M_{n-i+1}, that is, X_n is in $\mathrm{add}(M)$. This is a contradiction and shows that X_0 does not belong to $\mathrm{add}(M)$. Suppose $X_i \in \mathrm{add}(M)$ for some $0 < i < n$. Then there is an integer $j \in \{1, 2, \cdots, n\}$ such that X_i is a direct summand of M_j. Clearly, $i \neq j$, and there is an irreducible map from X_i to X_{j-1}. On the one hand, if $i > j$, then there is an irreducible map from $X_n = \tau^{n-i}X_i$ to $X_{n-i+j-1} = \tau^{n-i}X_{j-1}$. This implies that X_n is a direct summand of M_{n-i+j}, which is a contradiction. On the other hand, if $i < j$, then there is an irreducible map from $X_0 = \tau^{-i}X_i$ to $X_{j-1-i} = \tau^{-i}X_{j-1}$. It follows that X_0 is a direct summand of M_{j-i}. This is again a contradiction. Hence there is no X_i belonging to $\mathrm{add}(M)$.

Now let m be the minimal integer in $\{0, 1, \cdots, n\}$ such that $X_n \simeq X_m$. If $m = 0$, then $\mathrm{add}\,(M \oplus X_n) = \mathrm{add}\,(M \oplus X_0)$. This means that the endomorphism algebras $\mathrm{End}_A(M \oplus X_n)$ and $\mathrm{End}_A(M \oplus X_0)$ are Morita equivalent. Now we assume $m > 0$. Then the A-modules $X_0, X_1, \cdots, X_m$ are pairwise non-isomorphic. We consider the sequence

$$X_m \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow X_0.$$

Since $X_m \notin \mathrm{add}\,(M)$, every homomorphism from X_m to M factors through the map $X_m \longrightarrow M_m$ in the Auslander-Reiten sequence starting at X_m. This means that the map $X_m \longrightarrow M_m$ is a left $\mathrm{add}\,(M)$-approximation of X_m. Similarly, the map $M_1 \longrightarrow X_0$ is a right $\mathrm{add}\,(M)$-approximation of X_0. Let $V = M \oplus X_m$. Then $X_i \notin \mathrm{add}\,(V)$ for all $i = 0, 1, \cdots, m-1$. It follows that we have exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(V, X_i) \longrightarrow \mathrm{Hom}_A(V, M_i) \longrightarrow \mathrm{Hom}_A(V, X_{i-1}) \longrightarrow 0$$

for $i = 1, \cdots, m$. Connecting the above exact sequences, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(V, X_m) \longrightarrow \mathrm{Hom}_A(V, M_m) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(V, M_1) \longrightarrow \mathrm{Hom}_A(V, X_0).$$

This gives the first exact sequence in Lemma 3.4(2). The second exact sequence in Lemma 3.4(2) can be obtained similarly. Thus Corollary 3.11 follows immediately from Lemma 3.4. $\square$

Remark. In Corollary 3.11, if $X_n \notin \mathrm{add}\,(M)$ and $X_0, X_1, \cdots, X_n$ are pairwise non-isomorphic, then the tilting $\mathrm{End}\,(X \oplus M)$-module T defined in Lemma 3.4 has projective dimension n. Note that we always have $\mathrm{gl.dim}(\mathrm{End}_A(X \oplus M)) - n \le \mathrm{gl.dim}(\mathrm{End}_A(M \oplus Y)) \le \mathrm{gl.dim}(\mathrm{End}_A(X \oplus M)) + n$.

The following is another type of generalization of Corollary 3.6.

Proposition 3.12 *Let A be an Artin algebra.*

(1) *Suppose $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow Y_i \longrightarrow 0$ is an Auslander-Reiten sequence for $i = 1, 2, \cdots, n$. Let $X = \bigoplus_i X_i, M = \bigoplus_i M_i$ and $Y = \bigoplus_i Y_i$. If $\mathrm{add}\,(X) \cap \mathrm{add}\,(M) = 0 = \mathrm{add}\,(M) \cap \mathrm{add}\,(Y)$, then $\mathrm{End}_A(X \oplus M)$ and $\mathrm{End}_A(M \oplus Y)$ are derived-equivalent.*

(2) *Suppose $0 \longrightarrow X_1 \longrightarrow X_2 \oplus M_1 \longrightarrow Y_1 \longrightarrow 0$ and $0 \longrightarrow X_2 \longrightarrow Y_1 \oplus M_2 \longrightarrow Y_2 \longrightarrow 0$ are two Auslander-Reiten sequences such that neither X_2 is in $\mathrm{add}\,(M_1)$ nor Y_1 is in $\mathrm{add}\,(M_2)$. If $X_1 \notin \mathrm{add}\,(Y_1 \oplus M_2)$(or equivalently, $Y_2 \notin \mathrm{add}\,(X_2 \oplus M_1)$), then $\mathrm{End}_A(X_1 \oplus M_1 \oplus M_2)$ and $\mathrm{End}_A(M_1 \oplus M_2 \oplus Y_2)$ are derived-equivalent.*

Proof. (1) Under our assumption, the exact sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an almost add(M)-split sequence in A-mod. Therefore (1) follows from Theorem 3.5.

(2) There is an exact sequence

$$(*) \quad 0 \longrightarrow X_1 \longrightarrow M_1 \oplus M_2 \longrightarrow Y_2 \longrightarrow 0,$$

which can be constructed by the given two Auslander-Reiten sequences. Clearly, $X_1 \notin \mathrm{add}\,(X_2 \oplus M_1)$ since Auslander-Reiten quiver has no loops. By assumption, we see $X_1 \notin \mathrm{add}\,(M_1 \oplus M_2)$. Hence we can verify that the morphism $X_1 \longrightarrow M_1 \oplus M_2$ in $(*)$ is a left $\mathrm{add}\,(M_1 \oplus M_2)$-approximation of X_1. Similarly, we can see that the morphism $M_1 \oplus M_2 \longrightarrow Y_2$ in $(*)$ is a right $\mathrm{add}\,(M_1 \oplus M_2)$-approximation of Y_2. Thus $(*)$ is an almost $\mathrm{add}\,(M_1 \oplus M_2)$-split sequence in A-mod, and therefore the conclusion (2) follows from Theorem 3.5. $\square$

Remark. Usually, given two Auslander-Reiten sequences $0 \to X_i \to M_i \to Y_i \to 0$ $(1 \le i \le 2)$, we cannot get a derived equivalence between $\mathrm{End}_A(X_1 \oplus X_2 \oplus M_1 \oplus M_2)$ and $\mathrm{End}_A(M_1 \oplus M_2 \oplus Y_1 \oplus Y_2)$. For a counterexample, we refer the reader to Example 3 in the last section.

Now, we mention that, for an n-almost split sequence studied in [16], we have a statement similar to Corollary 3.11.

Proposition 3.13 *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of A-mod with A a finite-dimensional algebra over a field $(n \ge 1)$. Suppose X and Y are two indecomposable A-modules in $\mathcal{C}$ such that the sequence*

$$0 \longrightarrow X \xrightarrow{f} M_n \xrightarrow{t_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t_2} M_1 \xrightarrow{g} Y \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$. Then $\mathrm{End}_A(X \oplus \bigoplus_{i=1}^n M_i)$ and $\mathrm{End}_A(\bigoplus_{i=1}^n M_i \oplus Y)$ are derived-equivalent.

Proof. Let $M := \bigoplus_{i=1}^n M_i$. Suppose Y is a direct summand of some M_i. Then there is a canonical projection $\pi : M_i \longrightarrow Y$. Let $t_1 = g$ and $t_{n+1} = f$. We observe that all homomorphisms $t_1, \cdots, t_{n+1}$ are radical maps by the definition of an n-almost split sequence. Hence the composition $t_{i+1}\pi$ can not be a split epimorphism and consequently factors through $t_1 = g$, that is, $t_{i+1}\pi = u_1 g$ for a homomorphism $u_1 : M_{i+1} \longrightarrow M_1$. First, we assume that $i \neq n$. Then $t_{i+2}u_1 g = t_{i+2}t_{i+1}\pi = 0$. By [16, Theorem 2.5.3], we have $t_{i+2}u_1 = u_2 t_2$ for a homomorphism $u_2 : M_{i+2} \longrightarrow M_2$. Similarly, we get a homomorphism $u_k : M_{i+k} \longrightarrow M_k$ such that $t_{i+k}u_{k-1} = u_k t_k$ for $k = 2, 3, \cdots, n-i$. This allows us to form the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u_{n-i+1}"] & M_n \arrow[r, "t_n"] \arrow[d, "u_{n-i}"] & M_{n-1} \arrow[r] \arrow[d, "u_{n-i-1}"] & \cdots \arrow[r] & M_{i+1} \arrow[r, "t_{i+1}"] \arrow[d, "u_1"] & M_i \arrow[r, "t_i"] \arrow[d, "\pi"] & M_{i-1} \\
M_{n-i+1} \arrow[r, "t_{n-i+1}"] & M_{n-i} \arrow[r, "t_{n-i}"] & M_{n-i-1} \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r, "g"] & Y. &
\end{tikzcd}$$

Note that if $i = n$ then the above diagram still makes sense. We claim that u_{n-i+1} is a split monomorphism. If this is not the case, then the map u_{n-i+1} factors through f. This means that there is some map $h_n : M_n \longrightarrow M_{n-i+1}$ such that $fh_n = u_{n-i+1}$. Then we have $f(u_{n-i} - h_n t_{n-i+1}) = fu_{n-i} - u_{n-i+1}t_{n-i+1} = 0$. By [16, Theorem 2.5.3], there is some homomorphism $h_{n-1} : M_{n-1} \longrightarrow M_{n-i}$ such that $t_n h_{n-1} = u_{n-i} - h_n t_{n-i+1}$, that is, $u_{n-i} = t_n h_{n-1} + h_n t_{n-i+1}$. Similarly, we get $h_k : M_k \longrightarrow M_{k-i+1}$ such that $u_{k-i+1} = h_{k+1}t_{k-i+2} + t_{k-i+1}h$ for $k = n-2, n-3, \cdots, i$. Thus $t_{i+1}(\pi - h_i g) = t_{i+1}\pi - (u_{i+1} - h_{i+1}t_2)g = t_{i+1}\pi - u_{i+1}g = 0$. Hence $\pi - h_i g$ factors through t_i, say $\pi - h_i g = t_i h_{i-1}$. Then $\pi = h_i g + t_i h_{i-1}$, which is a radical map since both g and t_i are radical maps. This is a contradiction. Hence X is a direct summand of M_{n-i+1} and $\mathrm{add}\,(M \oplus X) = \mathrm{add}\,(M) = \mathrm{add}\,(M \oplus Y)$. Thus, $\mathrm{End}_A(M \oplus X)$ and $\mathrm{End}_A(M \oplus Y)$ are Morita equivalent.

Similarly, if X is a direct summand of some M_i, then Y is a direct summand of M_{n-i+1}. It follows that $\mathrm{End}_A(M \oplus X)$ and $\mathrm{End}_A(M \oplus Y)$ are Morita equivalent.

Now we assume that neither X nor Y is a direct summand of M. We use Lemma 3.4 to show Proposition 3.13. By a property of an n-almost split sequence (see [16, Theorem 2.5.3]) and the fact that X and Y do not lie in $\mathrm{add}(M)$, we see that f is a left $\mathrm{add}(M)$-approximation of X and g is a right $\mathrm{add}(M)$-approximation of Y. It remains to check the condition (2) in Lemma 3.4. However, it follows from [16, Theorem 2.5.3] (see Remark (3) at the end of the proof of Lemma 3.4) that we have two exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(V, X) \xrightarrow{(-,f)} \mathrm{Hom}_A(V, M_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(V, M_1) \xrightarrow{(-,g)} \mathrm{Hom}_A(V, Y),$$

$$0 \longrightarrow \mathrm{Hom}_A(Y, W) \xrightarrow{(g,-)} \mathrm{Hom}_A(M_1, W) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(M_n, W) \xrightarrow{(f,-)} \mathrm{Hom}_A(X, W)$$

for $V := X \oplus M$ and $W := M \oplus Y$. Thus the condition (2) in Lemma 3.4 is satisfied. Consequently, Proposition 3.13 follows from Lemma 3.4. $\square$

4 Auslander-Reiten sequences and BB-tilting modules

In this section, we point out that, when we restrict our consideration to Auslander-Reiten sequences, the tilting module defining the derived equivalence in Theorem 3.5 is of special form, namely it is a BB-tilting-module in the sense of Brenner and Butler [5]. This shows that the tilting theory and the Auslander-Reiten theory are so beautifully integrated with each other. We first recall the BB-tilting-module procedure in [5], and then give a generalization of a BB-tilting module, namely the notion of an n-BB-tilting module.

Let A be an Artin algebra and S a non-injective simple A-module with the following two properties: (a) $\mathrm{proj.dim}_A(\tau^{-1}S) \leq 1$, and (b) $\mathrm{Ext}^1_A(S, S) = 0$. Here τ^{-1} stands for the Auslander-Reiten translation TrD, and $\mathrm{proj.dim}_A(S)$ means the projective dimension of S. We denote the projective cover of S by $P(S)$, and assume that ${}_AA = P(S) \oplus P$ such that there is not any direct summand of P isomorphic to $P(S)$. Let $T = \tau^{-1}S \oplus P$. It is well-known that T is a tilting module. Such a tilting module is called a BB-*tilting module*. In particular, if S is a projective non-injective simple module, then T is automatically a BB-tilting module, this special case was first studied in [1], and the tilting module of this form is called an APR-*tilting module* in literature. Note that if S is a non-injective, projective simple A-module, then there is an Auslander-Reiten sequence

$$0 \longrightarrow S \longrightarrow P' \longrightarrow \tau^{-1}S \longrightarrow 0$$

in A-mod with P' projective.

Proposition 4.1 *Let A be an Artin algebra, and let $0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$ be an Auslander-Reiten sequence in A-mod. We define $V := M \oplus X$, $\Lambda = \mathrm{End}_A(V)$, $W = M \oplus Y$ and $\Gamma = \mathrm{End}_A(W)$. Then the derived equivalence between Λ and Γ in Theorem 3.5 is given by a BB-tilting module. In particular, if the Auslander-Reiten sequence*

$$0 \longrightarrow S \longrightarrow P' \longrightarrow \tau^{-1}S \longrightarrow 0$$

defines an APR-tilting module $T := P \oplus \tau^{-1}S$, then the sequence is an almost $\mathrm{add}(P)$-split sequence in A-mod and the derived equivalence between A and $\mathrm{End}_A(T)$ in Theorem 3.5 is given precisely by the APR-tilting module $T := P \oplus \tau^{-1}S$.

Proof. From the Auslander-Reiten sequence we have the following exact sequence

$$0 \to \mathrm{Hom}_A(V,X) \to \mathrm{Hom}_A(V,M) \xrightarrow{(-,g)} \mathrm{Hom}_A(V,Y).$$

Let L be the image of the map $(-,g)$. Then we have an exact sequence

$$(*) \quad 0 \to \mathrm{Hom}_A(V,X) \to \mathrm{Hom}_A(V,M) \xrightarrow{(-,g)} L \to 0.$$

(This is a minimal projective presentation of the Λ-module L). Let $T := L \oplus \mathrm{Hom}_A(V,M)$. Then T is the tilting module which defines the derived equivalence in Theorem 3.5. We shall show that T is a BB-tilting Λ-module. To prove this, it is sufficient to show that L is of the form $\tau^{-1}S$ for a simple Λ-module S.

If we apply $\mathrm{Hom}_\Lambda(-,\Lambda)$ to $(*)$, then we get an exact sequence of right Λ-modules:

$$\mathrm{Hom}_\Lambda(\mathrm{Hom}_A(V,M),\Lambda) \longrightarrow \mathrm{Hom}_\Lambda(\mathrm{Hom}_A(V,X),\Lambda) \longrightarrow \mathrm{Tr}_\Lambda(L) \longrightarrow 0,$$

which is isomorphic to the following exact sequence

$$\mathrm{Hom}_A(M,V) \xrightarrow{(f,-)} \mathrm{Hom}_A(X,V) \longrightarrow \mathrm{Tr}_\Lambda(L) \longrightarrow 0,$$

where Tr_Λ stands for the transpose over Λ. Note that the image of the map $(f,-)$ is the radical of the indecomposable projective right Λ-module $\mathrm{Hom}_A(X,V)$. Thus $\mathrm{Tr}_\Lambda(L)$ is a simple right Λ-module, and consequently, $\tau_\Lambda L$ is isomorphic to the socle S of the indecomposable injective Λ-module $D\mathrm{Hom}_A(X,V)$. Hence $L \simeq \tau_\Lambda^{-1}S$. Since X is not a direct summand of M, we see that $\mathrm{Ext}^1_\Lambda(S,S) = 0$. Thus T is a BB-tilting module. Note that if $X \not\simeq Y$ then $L \simeq \mathrm{Hom}_A(V,Y)$. In case of an APR-tilting module, we can see that the given Auslander-Reiten sequence is an almost $\mathrm{add}(P)$-split sequence. Thus Proposition 4.1 follows. $\square$

Now, we introduce the notion of an n-BB-tilting module: Let A be an Artin algebra. Recall that we denote by Ω^n the n-th syzygy operator, and by Ω^{-n} the n-th co-syzygy operator. As usual, D is the duality of an Artin algebra. Suppose S is a simple A-module and n is a positive integer. If S satisfies (a) $\mathrm{Ext}^j_A(D(A),S) = 0$ for all $0 \leq j \leq n-1$, and (b) $\mathrm{Ext}^i_A(S,S) = 0$ for all $1 \leq i \leq n$, we say that S defines an n-BB-tilting module, and that the module $T := \tau^{-1}\Omega^{-n+1}(S) \oplus P$ is an n-BB-*tilting module*, where P is the direct sum of all non-isomorphic indecomposable projective A-modules which are not isomorphic to $P(S)$, the projective cover of S. Note that (a) implies that the injective dimension of S is at least n and that the case $n = 1$ is just the usual BB-tilting module. The terminology is adjudged by the following lemma.

Lemma 4.2 *If S defines an n-BB-tilting A-module, then $T := \tau^{-1}\Omega^{-n+1}S \oplus P$ is a tilting module of projective dimension at most n.*

Proof. Let ν be the Nakayama functor $D\mathrm{Hom}_A(-, {}_AA)$. Suppose the sequence

$$0 \longrightarrow S \longrightarrow \nu P_0 \longrightarrow \nu P_1 \longrightarrow \cdots \longrightarrow \nu P_n \longrightarrow \cdots$$

is a minimal injective resolution of S with all P_i projective. Since $\mathrm{Ext}^i_A(D(A),S) = 0$ for $0 \leq i \leq n-1$, we have the following exact sequence by applying $\mathrm{Hom}_A(D(A),-)$ to the injective resolution:

$$0 \longrightarrow \mathrm{Hom}_A(D(A),S) \longrightarrow \mathrm{Hom}_A(D(A),\nu P_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(D(A),\nu P_n) \longrightarrow L \longrightarrow 0,$$

which is isomorphic to the following exact sequence

$$0 \longrightarrow 0 \longrightarrow P_0 \longrightarrow \cdots \longrightarrow P_n \longrightarrow L \longrightarrow 0.$$

This shows that $L \simeq \mathrm{TrD}\Omega_A^{-n+1}(S)$ and the projective dimension of L is at most n. Moreover, we have the following sequence:

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_A(L,P) \longrightarrow \mathrm{Hom}_A(P_n,P) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_0,P) \longrightarrow 0.$$

Since $\mathrm{Hom}_A(\nu P_j, \nu P) \simeq \mathrm{Hom}_A(P_j,P)$, we see that $(*)$ is isomorphic to the sequence

$$0 \longrightarrow \mathrm{Hom}_A(L,P) \longrightarrow \mathrm{Hom}_A(\nu P_n,\nu P) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(\nu P_0,\nu P) \longrightarrow 0,$$

which is exact because $\mathrm{Hom}_A(-,\nu P)$ is an exact functor. Note that $\mathrm{Hom}_A(S,\nu P)=0$ by the definition of P. This shows that $\mathrm{Ext}_A^i(L,P)=0$ for all $i>0$. Since $\mathrm{Ext}_A^i(S,S)=0$ for all $1\le i\le n$, this means that νP_0 is not a direct summand of νP_i for $1\le i\le n$. Thus $P(S)$ is not a direct summand of P_i for $1\le i\le n$, that is, $P_i\in \mathrm{add}\,(P)$ for all $1\le i\le n$. Now, if we apply $\mathrm{Hom}_A(L,-)$ to the projective resolution of L, we get $\mathrm{Ext}_A^{n+i}(L,P_0)\simeq \mathrm{Ext}_A^i(L,L)$ for all $i\ge 1$. Hence $\mathrm{Ext}_A^i(L,L)=0$ for all $i\ge 1$.

We note that $P_0=P(S)$ and there is an exact sequence

$$0 \longrightarrow A \longrightarrow P\oplus P_1 \longrightarrow \cdots \longrightarrow L \longrightarrow 0.$$

Altogether, we have shown that T is a tilting module of projective dimension at most n. $\square$

Proposition 4.3 (1) *Suppose $0\to X_i\to M_i\to X_{i-1}\to 0$ is an Auslander-Reiten sequence in* A-mod *for $i=1,2,\cdots,n$. Let $M=\bigoplus_{i=1}^n M_i$ and $V=M\oplus X_n$. If $X_n\notin \mathrm{add}\,(M)$ and if $X_0,X_1,\cdots,X_n$ are pairwise non-isomorphic, then the tilting* $\mathrm{End}_A(V)$*-module $T:=\mathrm{Hom}_A(V,X_0)\oplus \mathrm{Hom}_A(V,M)$ is an n-BB-tilting module.*

(2) *Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of* A-mod *with A a finite-dimensional algebra over a field $(n\ge 1)$. Suppose X and Y are two indecomposable A-modules in $\mathcal{C}$ such that the sequence*

$$0 \longrightarrow X \xrightarrow{f} M_n \xrightarrow{t_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t_2} M_1 \xrightarrow{g} Y \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$. We define $V=\bigoplus_{i=1}^n M_i\oplus X$, and L to be the image of the map $\mathrm{Hom}_A(V,g)$*. If $X\notin \mathrm{add}\,(\bigoplus_j M_j)$, then* $\mathrm{Hom}_A(V,M)\oplus L$ *is an n-BB-tilting* $\mathrm{End}_A(V)$*-module.*

Proof. The proof of (1) is similar to the one of Proposition 4.1. We leave it to the reader.

(2) We shall show that L is isomorphic to $\tau^{-1}\Omega_\Lambda^{-n+1}(S)$ with $S=\tau\Omega_\Lambda^{n-1}(L)$ being a simple Λ-module. It is easy to see that $D(S)=\mathrm{Tr}\Omega_\Lambda^{n-1}(L)$ is a simple right Λ-module. In fact, it is isomorphic to the top of the indecomposable right Λ-module $\mathrm{Hom}_A(X,V)$, and is not injective since $X\notin \mathrm{add}\,(\bigoplus_j M_j)$. Further, it follows from $X\notin \mathrm{add}\,(\bigoplus_i M_i)$ that we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(Y,V) \longrightarrow \mathrm{Hom}_A(M_1,V) \longrightarrow \mathrm{Hom}_A(M_2,V) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(M_n,V)$$

$$\longrightarrow \mathrm{Hom}_A(X,V) \longrightarrow \mathrm{Tr}\Omega_\Lambda^{n-1}(L)=D(S) \longrightarrow 0.$$

If we apply $\mathrm{Hom}_\Lambda(-,\Lambda)$ to this sequence, we can see that $\mathrm{Ext}_\Lambda^i(D(S),\Lambda)=0$ for all $0\le i\le n-1$. This is just the condition (a) in the definition of an n-BB-tilting module. To see that $\mathrm{Ext}_\Lambda^i(S,S)=0$ for all $1\le i\le n$, we show that $\mathrm{Ext}_{\Lambda^{op}}^i(D(S),D(S))=0$ for all $1\le i\le n$. This means that the projective cover $\mathrm{Hom}_A(X,V)$ of the right Λ-module $D(S)$ is not a direct summand of $\mathrm{Hom}_A(M_i,V)$ for all $1\le i\le n$. However, this follows from the assumption that $X\notin \mathrm{add}\,(\bigoplus_{j=1}^n M_j)$. Thus the condition (b) of an n-BB-tilting module is fulfilled. $\square$

Remarks. (1) One can see that a non-injective simple A-module S defines an n-BB-tilting module if and only if (a') $\mathrm{proj.dim}_A(\tau^{-1}\Omega^{-n+1}(S))\le n$, (b') $\mathrm{Ext}_A^i(S,S)=0$ for all $1\le i\le n$ and (c') $\mathrm{Ext}_A^i(D(A),S)=0$ for all $1\le i\le n-1$. Note tat if a simple module S defines an n-BB-tilting module then the injective dimension of S is n if and only if $\mathrm{Hom}_A(\tau^{-1}\Omega^{-n+1}(S),A)=0$.

(2) With the same method as in Proposition 4.3, we can prove the following fact:

Let $\mathcal{C}$ be a maximal $(n-1)$-orthogonal subcategory of A-mod with A a finite-dimensional algebra over a field $(n\ge 1)$. Suppose X and Y are two indecomposable A-modules in $\mathcal{C}$ such that the sequence

$$0 \longrightarrow X \xrightarrow{f} M_n \xrightarrow{t_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{t_2} M_1 \xrightarrow{g} Y \longrightarrow 0$$

is an n-almost split sequence in $\mathcal{C}$. We define $M=\bigoplus_{i=1}^n M_i$, $V=M\oplus X$, and $U=X\oplus M\oplus Y$. Let Σ be the endomorphism algebra of U. If $X\notin \mathrm{add}\,(M\oplus Y)$, then $T:=\mathrm{Hom}_A(V,U)\oplus S^X$ is an (n+1)-BB-tilting

right Σ-module, where S^X is the top of the right Σ-module $\mathrm{Hom}_A(X,U)$. If we define $\Delta = \mathrm{End}\,(T_\Sigma)$, then $\mathrm{Hom}_\Sigma(\mathrm{Hom}_A(V,U)_\Sigma, T_\Sigma) \oplus \mathrm{Hom}_\Sigma(\mathrm{Hom}_A(Y,U)_\Sigma, T_\Sigma)$ is an $(n+1)$-APR-tiling Δ-module, that is, it is an $(n+1)$-BB-tilting Δ-module defined by the projective simple Δ-module $\mathrm{Hom}_\Sigma(S^X, T)$. Note that Δ is a one-point extension of $\mathrm{End}_A(V)$ because $\mathrm{Hom}_\Sigma(S^X, \Sigma) = 0$.

5 Auslander-Reiten triangles and derived equivalences

By Corollary 3.6, one can get a derived equivalence from an Auslander-Reiten sequence. An analogue of an Auslander-Reiten sequence in a triangulated category is the notion of Auslander-Reiten triangle. Thus, a natural question rises: is it possible to get a derived equivalence from an Auslander-Reiten triangle in a triangulated category? In this section, we shall discuss this question. First, let us briefly recall some basic definitions concerning Auslander-Reiten triangles. For more details, we refer the reader to [11].

Let R be a commutative ring. Let $\mathcal{C}$ be a triangulated R-category such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has finite length as an R-module for all X and Y in $\mathcal{C}$. In this case, we say that $\mathcal{C}$ is a Hom-finite triangulated R-category. Suppose further that the category $\mathcal{C}$ is a Krull-Schmidt category. A triangle $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{w} X[1]$ in $\mathcal{C}$ is called an *Auslander-Reiten triangle* if

(AR1) X and Y are indecomposable;
(AR2) $w \neq 0$;
(AR3) if $t: U \longrightarrow Y$ is not a split epimorphism, then $tw = 0$.

Note that neither f is a monomorphism nor g is an epimorphism in an Auslander-Reiten triangle. This is a difference of an Auslander-Reiten triangle from an almost $\mathcal{D}$-split sequence. Thus, an Auslander-Reiten triangle in a triangulated category may not be an almost $\mathcal{D}$-split sequence. Also, an Auslander-Reiten sequence in the module category of an Artin algebra in general may not give us an Auslander-Reiten triangle in its derived module category. For an Artin algebra, we even don't know whether its stable module category has a triangulated structure except that the Artin algebra is self-injective. In this case, an Auslander-Reiten sequence can be extended to an Auslander-Reiten triangle in the stable module category.

Recall that a morphism $f: U \longrightarrow V$ in a category $\mathcal{C}$ is called a *split monomorphism* if there is a morphism $g: V \longrightarrow U$ in $\mathcal{C}$ such that $fg = id_U$; a *split epimorphism* if $gf = id_V$; and an *irreducible* morphism if f is neither a split monomorphism nor a split epimorphism, and, for any factorization $f = f_1 f_2$ in $\mathcal{C}$, either f_1 is a split monomorphism or f_2 is a split epimorphism.

Suppose $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{w} X[1]$ is an Auslander-Reiten triangle in a triangulated category $\mathcal{C}$. Then we have the following basic properties:

(1) $fg = 0$ and $gw = 0$. Moreover, both f and g are irreducible morphisms.

(2) If $s: X \to U$ is not a split monomorphism, then s factors through f. Similarly, if $t: V \to Y$ is not a split epimorphism, then t factors through g.

(3) Let V be an indecomposable object in $\mathcal{C}$. Then V is a direct summand of M if and only if there is an irreducible map from V to Y if and only if there is an irreducible map from X to V.

We mention that in any triangulated category $\mathcal{C}$ the functors $\mathrm{Hom}_{\mathcal{C}}(V,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,V)$ are cohomological functors for each object $V \in \mathcal{C}$ (see [11, Proposition 1.2, p.4]).

The following is an expected result for Auslander-Reiten triangles.

Proposition 5.1 *Let $\mathcal{C}$ be a Hom-finite, Krull-Schmidt, triangulated R-category. Suppose $X \xrightarrow{f} M \xrightarrow{g} Y \xrightarrow{w} X[1]$ is an Auslander-Reiten triangle in $\mathcal{C}$ such that $X[1] \notin \mathrm{add}\,(M \oplus Y)$. If N is an object in $\mathcal{C}$ such that none of $X, Y, X[1]$ and $Y[-1]$ belongs to $\mathrm{add}\,(N)$, then $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus Y)$ are derived-equivalent via a tilting module. In particular, $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ are derived-equivalent via a tilting module.*

Proof. First, if X is a direct summand of M, then there is an irreducible map from X to Y. It follows from the property (3) of an Auslander-Reiten triangle that Y is a direct summand of M. Similarly, if Y is a direct summand of M, then so is X. Thus, if X or Y is in $\mathrm{add}\,(M)$, then $\mathrm{add}\,(N \oplus M \oplus X) = \mathrm{add}\,(N \oplus M \oplus Y) = \mathrm{add}\,(N \oplus M)$. In this case, both $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus Y)$ are Morita equivalent to $\mathrm{End}_{\mathcal{C}}(N \oplus M)$, and therefore $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus X)$ and $\mathrm{End}_{\mathcal{C}}(N \oplus M \oplus Y)$ are derived-equivalent. Now, we assume that neither X nor Y is in $\mathrm{add}\,(M)$. For simplicity, we set $U := N \oplus M$, $V := U \oplus X$ and $W := U \oplus Y$.

Denote by Λ the endomorphism ring of V. Since X and Y are not in add(U), we see that f is a left add(U)-approximation of X and g is a right add(U)-approximation of Y. To see that the condition (2) in Lemma 3.4 is satisfied, we consider the exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(V, M[-1]) \xrightarrow{\delta} \mathrm{Hom}_{\mathcal{C}}(V, Y[-1]) \to \mathrm{Hom}_{\mathcal{C}}(V, X) \to \mathrm{Hom}_{\mathcal{C}}(V, M) \to \mathrm{Hom}_{\mathcal{C}}(V, Y).$$

We have to show that the map δ is surjective. By assumption, we have $Y[-1] \notin \mathrm{add}\,(N)$ and $Y[-1] \not\simeq X$ since $Y \not\simeq X[1]$. If $Y[-1] \in \mathrm{add}\,(M)$, then there is an irreducible map from X to $Y[-1]$ by the property (3), and therefore there is an irreducible map from $X[1]$ to Y. It follows that $X[1]$ is a direct summand of M, which contradicts to our assumption that $X[1] \notin \mathrm{add}\,(M)$. This shows that $Y[-1] \notin \mathrm{add}\,(M)$. Thus any morphism from V to $Y[-1]$ cannot be a split epimorphism. This implies that the map δ is surjective by the property (2) of an Auslander-Reiten triangle since the triangle $X[-1] \longrightarrow M[-1] \longrightarrow Y[-1] \longrightarrow X$ is also an Auslander-Reiten triangle. Hence we have a desired exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(V, Y).$$

Similarly, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, W) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, W) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, W).$$

Thus Proposition 5.1 follows from Lemma 3.4 by taking $n = 1$. $\square$

From Proposition 5.1 we get the following corollary.

Corollary 5.2 *Let A be a self-injective Artin algebra. Suppose $0 \to X \to M \to Y \to 0$ is an Auslander-Reiten sequence such that $\Omega^{-1}(X) \notin \mathrm{add}\,(M \oplus Y)$. Then $\underline{\mathrm{End}}_A(M \oplus X)$ and $\underline{\mathrm{End}}_A(M \oplus Y)$ are derived-equivalent, where $\underline{\mathrm{End}}_A(M)$ stands for the quotient of $\mathrm{End}_A(M)$ by the ideal of those endomorphisms of M, which factor through a projective A-module.*

Proof. If A is a self-injective Artin algebra, then every Auslander-Reiten sequence $0 \to X \to M \to Y \to 0$ in A-mod can be extended to an Auslander-Reiten triangle

$$X \longrightarrow M \longrightarrow Y \longrightarrow \Omega_A^{-1} X$$

in the triangulated category A-$\underline{\mathrm{mod}}$ which is equivalent to $\mathscr{D}^b(A)/\mathscr{K}^b(A)$ (for details, see [11]). Thus Corollary 5.2 follows. $\square$

Note that under the assumptions in Proposition 5.1 the corresponding statement of Proposition 4.1 holds true for an Auslander-Reiten triangle.

Finally, let us remark that Corollary 5.2 may fail if A is not self-injective; for example, if we take A to be the path algebra (over a field k) of the quiver $2 \longrightarrow 1 \longleftarrow 3$, then there is an Auslander-Reiten sequence

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(3) \longrightarrow I(1) \longrightarrow 0,$$

where $P(i)$ and $I(i)$ stand for the projective and injective modules corresponding to the vertex i, respectively. Clearly, this is a desired counterexample.

6 An Example

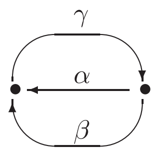
In this section, we illustrate our results with an example.

Example 1. Let k be a field, and let $A = k[x, y]/(x^2, y^2)$. If Y denotes the simple A-module, then there is an Auslander-Reiten sequence

$$0 \longrightarrow X \longrightarrow N \oplus N \longrightarrow Y \longrightarrow 0$$

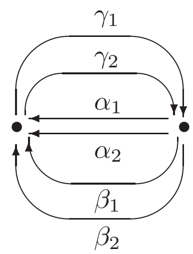
in A-mod. Note that $X = \Omega_A^2(Y)$ and N is the radical of A. By Theorem 1.1 or Corollary 1.2, the two algebras $\mathrm{End}_A(N \oplus Y)$ and $\mathrm{End}_A(N \oplus X)$ are derived-equivalent. Though the local diagram of the Auslander-Reiten sequence is reflectively symmetric, the two algebras $\mathrm{End}_A(N \oplus Y)$ and $\mathrm{End}_A(N \oplus X)$ are very different. This can be seen by the following presentations of the two algebras given by quiver with relations:

$\mathrm{End}_A(N \oplus Y)$



$\alpha\gamma = 0 = \beta\gamma.$

$\mathrm{End}_A(N \oplus X)$



$$\gamma_i\alpha_j = 0 = \gamma_i\beta_j, i \neq j,$$
$$\gamma_1\beta_1 = \gamma_2\beta_2, \quad \gamma_1\alpha_1 = \gamma_2\alpha_2,$$
$$\alpha_1\gamma_2 = \beta_1\gamma_1, \quad \alpha_2\gamma_2 = \beta_2\gamma_1.$$

Note that the algebra $\mathrm{End}_A(N \oplus Y)$ is a 7-dimensional algebra of global dimension 2, while the algebra $\mathrm{End}_A(N \oplus X)$ is a 19-dimensional algebra of global dimension 3. Hence the two algebras are not stably equivalent of Morita type since global dimension is invariant under stable equivalences of Morita type (see [23]). A calculation shows that the Cartan determinants of the both algebras equal 1.

Recall that the Cartan matrix of an Artin algebra A is defined as follows: Let $S_1, \cdots, S_n$ be a complete list of non-isomorphic simple A-modules, and let P_i be a projective cover of S_i. We denote the multiplicity of S_j in P_i as a composition factor by $[P_i : S_j]$. The Catan matrix of A is the $n \times n$ matrix $([P_i : S_j])_{1 \leq i,j \leq n}$, and its determinant is called the *Cartan determinant* of A. It is well-known that the Cartan determinant is invariant under derived equivalences.

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